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Mapping of the five-parameter exponential-type potential model into trigonometric-type potentials

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Abstract

We choose the five-parameter exponential-type potential model as input, and construct five trigonometric-type potentials via point canonical transformations. Their energy spectra and wavefunctions are obtained in a unified manner by using the expressions for the energy spectra and wavefunctions of the five-parameter exponential-type potential model.

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1. Introduction

The study of spectral problems for exactly solvable potentials has been the subject of many investigations. Many exactly solvable potentials are hyperbolic or trigonometric functions of the spatial coordinate [1, 2]. These solvable potentials are widely used in physics [3]. For the solvable hyperbolic-type and trigonometric-type potentials with shape invariance [1], the corresponding Schrödinger equation can be reduced to a hypergeometric equation. De *et al* [4] studied the inter-relation for five hyperbolic-type potentials and three trigonometric-type potentials via point canonical transformations (PCT). PCT have been studied in the path integral approach to quantum mechanical problems [5]. Using PCT and shape-invariant potentials as input, Dutt *et al* [6] obtained a new class of one-dimensional conditionally exactly solvable potentials. Kocak *et al* [7] applied PCT to inter-connect non-central potentials among themselves for mapping purposes.

In 2002, we [8] proposed an exactly solvable five-parameter exponential-type potential model (FPEPM), which contains five exactly solvable hyperbolic-type potentials discussed by De *et al* in [4], i.e., the Rosen–Morse well, Eckart, Scarf II, generalized Pöschl–Teller and Pöschl–Teller II potentials. Their complex versions with PT symmetry [9] and pseudo-Hermiticity [10] are also included in FPEPM as special cases [11]. For the corresponding trigonometric-type potentials, their complex versions have been studied in detail by Lévai and Znojil [12]. Recently, Ramazan and Mehmet [13] constructed the five trigonometric-type potentials with the help of a general form of the generators of $su(1,1)$ algebra. However, they gave only closed analytic expressions for the energy spectra, and did not give the wavefunction expression in a closed form. The aim of this paper is to obtain trigonometric-type potentials from the stated FPEPM and point canonical transformation, in order to see if the method can reproduce the known solutions for these trigonometric-type potentials. We do this in this paper to obtain closed analytic expressions both for the energy spectra and wavefunctions. The wavefunction expression for the Scarf I potential given in the present work is, to the best of our knowledge, not stated in the published literature. The inter-relations presented in this paper are useful for a unified treatment of the exponential-type potentials and the corresponding trigonometric-type potentials.

The arrangement of this paper is as follows. In section 2, we review briefly PCT in non-relativistic quantum mechanics. In section 3, we give a brief survey of FPEPM. In section 4, we construct five trigonometric-type potentials by using FPEPM and the method of PCT. Finally, we give some concluding remarks in section 5.

2. Point canonical transformations

In this section, we review briefly the method of PCT [4]. For a one-dimensional potential $V(x)$, the Schrödinger equation is

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\Psi(x) = E\Psi(x). \tag{1}$$

Throughout this paper we take natural units, $2m = \hbar = 1, \hbar = \frac{h}{2\pi}$, h is the Planck constant. Invoking a transformation $x \to u$ through a mapping function

$$x = f(u), \tag{2}$$

and rewriting the wavefunction in the form

$$\Psi(x) = \sqrt{\frac{\mathrm{d}f(u)}{\mathrm{d}u}}\tilde{\Psi}(u), \tag{3}$$

we obtain a transformed Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \tilde{V}(u)\right)\tilde{\Psi}(u) = \tilde{E}\tilde{\Psi}(u), \tag{4}$$

where the new potential $\tilde{V}(u)$ is given by

$$\tilde{V}(u) - \tilde{E} = f'^2[V(f(u)) - E] + \frac{1}{2}\left[\frac{3}{2}\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right], \tag{5}$$

where the prime denotes differentiation with respect to the variable u.

Putting an exactly solvable potential as $V(x)$ and choosing a proper mapping function $f(u)$, one can obtain a new analytically solvable potential $\tilde{V}(u)$, the energy spectrum and wavefunction of which can be expressed in a closed form. In the present work, we choose FPEPM as the old potential $V(x)$ and construct five trigonometric-type potentials by using the method of PCT.

3. Five-parameter exponential-type potential model

In [8], we proposed FPEPM, which can be expressed in the form of [14]

$$V(x) = -\frac{g}{2q}\tanh_q \alpha x - \gamma\,\mathrm{sech}_q^2\,\alpha x + \frac{gQ_3}{2Q_2}\mathrm{sech}_q\,\alpha x - \eta\,\mathrm{sech}_q\,\alpha x \cdot \tanh_q \alpha x, \tag{6}$$

where the range of parameter q is $q > 0$ for $-\infty < x < \infty$, and $-1 \leqslant q < 0$ or $q > 0$ for $0 \leqslant x < \infty$. The parameters Q_2 and Q_3 are functions of γ and η. Their relationships are given by

$$Q_2^2 - qQ_3^2 - 2\alpha q Q_2 = 4q\gamma, \tag{7}$$

$$Q_2 Q_3 - \alpha q Q_3 = 2q\eta. \tag{8}$$

In equation (6), we have applied the deformed hyperbolic functions, which are defined by [15]

$$\begin{aligned}
&\sinh_q x = \frac{e^x - q\,e^{-x}}{2}, \qquad \cosh_q x = \frac{e^x + q\,e^{-x}}{2}, \qquad \mathrm{sech}_q\,x = \frac{1}{\cosh_q x},\\
&\mathrm{cosech}_q\,x = \frac{1}{\sinh_q x}, \qquad \tanh_q \alpha x = \frac{\sinh_q \alpha x}{\cosh_q \alpha x}, \qquad \coth_q \alpha x = \frac{\cosh_q \alpha x}{\sinh_q \alpha x},
\end{aligned} \tag{9}$$

where $q > 0$ is a real parameter. When q is complex, we call the above deformed hyperbolic functions the generalized deformed hyperbolic functions. Applying the deformed hyperbolic functions, some exact solutions of the multi-component nonlinear Schrödinger and Klein–Gordon equations have been obtained [16]. With the help of the deformed hyperbolic functions, some deformed hyperbolic molecular potentials [17] and pseudo-Hermitian potentials [11] are also constructed. Recently, de Lima and de Lima Rodrigues [18] used the deformed hyperbolic functions in the stability equation, and constructed the deformed topological kink associated with the deformed ϕ^4 potential model.

When $Q_3 = 0$, the allowed value of the energy E for the potential (6) is given by [14]

$$E_n = -\left(\frac{g}{4\alpha q}\right)^2 \frac{1}{\left(n + \frac{1}{2} - \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}}\right)^2} - \alpha^2\left(n + \frac{1}{2} - \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}}\right)^2, \tag{10}$$

where the quantum number n satisfies the restriction, $n < \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}} - \frac{1}{2}$, $n = 0, 1, 2, \ldots$. The corresponding unnormalized wavefunction is expressed in terms of Jacobi polynomials as [14]

$$\Psi_n(x) = \frac{\Gamma(-2p)\Gamma(n)}{\Gamma(n-2p)}\left(\frac{1+\tanh_q \alpha x}{2}\right)^{-p}\left(\frac{1-\tanh_q \alpha x}{2}\right)^{-r} P_n^{-2p,\,-2r}(-\tanh_q \alpha x), \tag{11}$$

where

$$p = \frac{1}{2}\left[n + \frac{1}{2} - \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}} + \frac{g}{4q\alpha^2}\frac{1}{\left(n + \frac{1}{2} - \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}}\right)}\right], \tag{12}$$

$$r = \frac{1}{2}\left[n + \frac{1}{2} - \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}} - \frac{g}{4q\alpha^2}\frac{1}{\left(n + \frac{1}{2} - \sqrt{\frac{\gamma}{q\alpha^2} + \frac{1}{4}}\right)}\right]. \tag{13}$$

When $g = 0$, the energy spectrum for the potential (6) is given by [14]

$$E_n = -\alpha^2 \left[-n - \frac{1}{2} + \frac{1}{2} \left(\sigma \sqrt{\frac{1}{4} + \frac{\gamma}{q\alpha^2} + \frac{\eta}{\mathrm{i}\alpha^2 q^{1/2}}} + \tau \sqrt{\frac{1}{4} + \frac{\gamma}{q\alpha^2} - \frac{\eta}{\mathrm{i}\alpha^2 q^{1/2}}} \right) \right]^2, \tag{14}$$

where $\sigma = \pm 1$ and $\tau = \pm 1$. The unnormalized wavefunction for the potential (6) is given by [14]

$$\Psi_n(x) = \frac{1}{(\cosh_q \alpha x)^{p+r}} \exp((p - r) \tanh^{-1}(\mathrm{i}q^{-1/2} \sinh_q \alpha x))$$
$$\times P_n^{-2p-\frac{1}{2},\, -2r-\frac{1}{2}} (\mathrm{i}q^{-1/2} \sinh_q \alpha x), \tag{15}$$

where

$$p = -\frac{1}{4} + \frac{\sigma}{2} \sqrt{\frac{1}{4} + \frac{\gamma}{q\alpha^2} + \frac{\eta}{\mathrm{i}\alpha^2 q^{1/2}}}, \tag{16}$$

$$r = -\frac{1}{4} + \frac{\tau}{2} \sqrt{\frac{1}{4} + \frac{\gamma}{q\alpha^2} - \frac{\eta}{\mathrm{i}\alpha^2 q^{1/2}}}. \tag{17}$$

The condition of the normalizability of the function (15) limits the range of admissible quantum numbers n via

$$n < \mathrm{Re} \left[\frac{1}{2} \left(\sigma \sqrt{\frac{1}{4} + \frac{\gamma}{q\alpha^2} + \frac{\eta}{\mathrm{i}\alpha^2 q^{1/2}}} + \tau \sqrt{\frac{1}{4} + \frac{\gamma}{q\alpha^2} - \frac{\eta}{\mathrm{i}\alpha^2 q^{1/2}}} \right) \right] - \frac{1}{2}.$$

If we take $Q_3 = 0$, we obtain $\eta = 0$ from equation (8). Considering the above restriction condition for the quantum number n, both of σ and τ can only be taken ± 1. Thus, in the case of $g = 0$ and $Q_3 = 0$, equation (10) is identical to equation (14).

4. Mapping of FPEPM into trigonometric-type potentials

In this section, we construct some trigonometric-type potentials by using the method of PCT. We also investigate the energy spectra and wavefunctions for the trigonometric-type potentials by using the expressions for the energy spectra and wavefunctions of FPEPM. In our scheme, first, we choose the particular values of the parameters in FPEPM expressed in equation (6). Second, invoking a transformation of the independent variable $x = f(\theta)$, we take the potential (6) as the old potential and express it in the variable θ. Third, using equation (5), we obtain the new potential with trigonometric-type, and get the relations among the parameters by comparing the coefficients of the corresponding terms in the expressions of the old potential and new potential. Fourth, using the energy spectrum expression of the old potential and the relations among the parameters, we obtain the energy eigenvalues of the new potential. Fifth, with the help of the wavefunction expression for the old potential and the relations among the parameters, we get the wavefunction expression for the new potential. Finally, we compare our results with those obtained in other methods.

4.1. Rosen–Morse I potential

Under the condition $-\frac{\pi}{2} \leqslant \alpha\theta \leqslant \frac{\pi}{2}$, using the transformation $x \to \theta$ through a mapping function

$$x \equiv f(\theta) = \frac{1}{\alpha} \ln \left(\frac{1 - \sin \alpha\theta}{\cos \alpha\theta} \right), \tag{18}$$

and choosing $g = 0$, $q = 1$, FPEPM expressed in equation (6) becomes

$$V(x) = -\gamma\cos^2\alpha\theta + \eta\sin\alpha\theta\cos\alpha\theta. \tag{19}$$

Substituting equations (18) and (19) into (5), we obtain

$$\tilde{V} - \tilde{E} = -\gamma - \frac{\alpha^2}{4} + \left(-E - \frac{\alpha^2}{4}\right)\sec^2\alpha\theta + \eta\tan\alpha\theta. \tag{20}$$

For the Rosen–Morse I potential, we write $\tilde{V}$ as

$$\tilde{V} = -A^2 + \frac{B^2}{A^2} + A(A-\alpha)\sec^2\alpha\theta + 2B\tan\alpha\theta. \tag{21}$$

Comparing the coefficients of the corresponding terms in equations (20) and (21), we have

$$-\gamma - \frac{\alpha^2}{4} + \tilde{E} = -A^2 + \frac{B^2}{A^2}, \tag{22}$$

$$-E - \frac{\alpha^2}{4} = A(A-\alpha), \tag{23}$$

$$\eta = 2B. \tag{24}$$

Using the expression for (14), solving equations (23) and (24) yields

$$\gamma = \frac{(A+n\alpha)^4 - B^2}{(A+n\alpha)^2} - \frac{\alpha^2}{4}. \tag{25}$$

Substituting (25) into (22), we obtain the energy spectra for the Rosen–Morse I potential,

$$\tilde{E}_n = -A^2 + \frac{B^2}{A^2} + (A+n\alpha)^2 - \frac{B^2}{(A+n\alpha)^2}. \tag{26}$$

Replacing η and γ in equations (16) and (17) by the expressions given in equations (24) and (25), respectively, we obtain

$$p = -\frac{1}{4} + \frac{1}{2}\left[\frac{A+n\alpha}{\alpha} - \frac{\mathrm{i}B}{\alpha(A+n\alpha)}\right], \tag{27}$$

$$r = -\frac{1}{4} + \frac{1}{2}\left[\frac{A+n\alpha}{\alpha} + \frac{\mathrm{i}B}{\alpha(A+n\alpha)}\right]. \tag{28}$$

Substituting the above expressions into equation (15) and using the wavefunction transformation (3) and coordinate transformation (18), we obtain the unnormalized eigenfunctions of the Rosen–Morse I potential,

$$\tilde{\Psi}_n(\theta) = (1+\tan^2\alpha\theta)^{-\frac{A+n\alpha}{2\alpha}}\exp\left(-\frac{B}{A+n\alpha}\theta\right)P_n^{-\frac{A+n\alpha}{\alpha}+\frac{\mathrm{i}B}{\alpha(A+n\alpha)},\,-\frac{A+n\alpha}{\alpha}-\frac{\mathrm{i}B}{\alpha(A+n\alpha)}}(-\mathrm{i}\tan\alpha\theta). \tag{29}$$

This result coincides completely with the result given by De *et al* in table 2 of [4].

4.2. Trigonometric-type Eckart potential

Invoking a transformation of the independent variable

$$x \equiv f(\theta) = \frac{1}{\alpha}\ln\left(\frac{1-\cos\alpha\theta}{\sin\alpha\theta}\right), \tag{30}$$

and choosing $g = 0$, $q = 1$, FPEPM expressed in equation (6) becomes

$$V(x) = -\gamma\sin^2\alpha\theta + \eta\sin\alpha\theta\cos\alpha\theta. \tag{31}$$

The parameter range of θ is $0 \leqslant \alpha\theta \leqslant \pi$. Substituting equations (30) and (31) into (5), we obtain

$$\tilde{V} - \tilde{E} = -\gamma - \frac{\alpha^2}{4} + \left(-E - \frac{\alpha^2}{4}\right) \operatorname{cosec}^2 \alpha\theta + \eta \cot \alpha\theta. \tag{32}$$

We consider the trigonometric-type Eckart potential, and write $\tilde{V}$ as

$$\tilde{V} = -A^2 + \frac{B^2}{A^2} + A(A+\alpha) \operatorname{cosec}^2 \alpha\theta - 2B \cot \alpha\theta. \tag{33}$$

Comparing the coefficients of the corresponding terms in equations (32) and (33), we have

$$-\gamma - \frac{\alpha^2}{4} + \tilde{E} = -A^2 + \frac{B^2}{A^2}, \tag{34}$$

$$-E - \frac{\alpha^2}{4} = A(A+\alpha), \tag{35}$$

$$\eta = -2B. \tag{36}$$

With the help of expression (14), solving equations (35) and (36), we obtain

$$\gamma = \frac{(A-n\alpha)^4 - B^2}{(A-n\alpha)^2} - \frac{\alpha^2}{4}. \tag{37}$$

Substituting the expression of γ given in equation (37) into (34), we obtain the energy spectra for the trigonometric-type Eckart potential,

$$\tilde{E}_n = -A^2 + \frac{B^2}{A^2} + (A-n\alpha)^2 - \frac{B^2}{(A-n\alpha)^2}. \tag{38}$$

From equations (16) and (17), we obtain

$$p = -\frac{1}{4} + \frac{1}{2}\left[\frac{A-n\alpha}{\alpha} + \frac{\mathrm{i}B}{\alpha(A-n\alpha)}\right], \tag{39}$$

$$r = -\frac{1}{4} + \frac{1}{2}\left[\frac{A-n\alpha}{\alpha} - \frac{\mathrm{i}B}{\alpha(A-n\alpha)}\right]. \tag{40}$$

Substituting the above expressions into equation (15) and using the coordinate transformation (30), we obtain the eigenfunctions of the trigonometric-type Eckart potential,

$$\tilde{\Psi}_n(\theta) = (\sin \alpha\theta)^{\frac{A-n\alpha}{\alpha}} \exp\left(\frac{B}{A-n\alpha}\theta\right) P_n^{\frac{A}{\alpha}-n+\mathrm{i}\frac{B}{\alpha(A-n\alpha)},\, \frac{A}{\alpha}-n-\mathrm{i}\frac{B}{\alpha(A-n\alpha)}}(-\mathrm{i}\cot \alpha\theta). \tag{41}$$

The results given in equations (38) and (41) are consistent with the results given by Lévai in table 1 of [2].

4.3. Scarf I potential

In the case of $-\frac{\pi}{2} \leqslant \alpha\theta \leqslant \frac{\pi}{2}$, the Scarf I potential can be constructed by introducing a mapping function

$$x \equiv f(\theta) = \frac{1}{\alpha} \ln\left(\frac{1 - \sin \alpha\theta}{\cos \alpha\theta}\right), \tag{42}$$

and putting $Q_3 = 0$, $q = 1$. Making the corresponding replacements for the parameters in equation (6), FPEPM takes the form

$$V(x) = \frac{g}{2} \sin \alpha\theta - \gamma \cos^2 \alpha\theta. \tag{43}$$

Substituting equations (42) and (43) into (5), we obtain

$$\tilde{V} - \tilde{E} = -\gamma - \frac{\alpha^2}{4} + \left(-E - \frac{\alpha^2}{4}\right)\sec^2\alpha\theta + \frac{g}{2}\sec\alpha\theta\tan\alpha\theta. \tag{44}$$

We consider the Scarf I potential, which takes the form

$$\tilde{V} = -A^2 + (A^2 + B^2 - \alpha A)\sec^2\alpha\theta - B(2A - \alpha)\sec\alpha\theta\tan\alpha\theta. \tag{45}$$

Comparing equations (44) and (45), we have

$$-\gamma - \frac{\alpha^2}{4} + \tilde{E} = -A^2, \tag{46}$$

$$-E - \frac{\alpha^2}{4} = A^2 + B^2 - \alpha A, \tag{47}$$

$$\frac{g}{2} = -B(2A - \alpha). \tag{48}$$

Using the energy spectrum expression (10), we solve equations (47) and (48), and get

$$\gamma = (A + n\alpha)^2 - \frac{\alpha^2}{4}. \tag{49}$$

Substituting (49) into (46), we obtain the energy spectra for the Scarf I potential,

$$\tilde{E}_n = -A^2 + (A + n\alpha)^2. \tag{50}$$

Substituting the expressions for g and γ given in equations (48) and (49) into equations (12) and (13), we obtain

$$p = \frac{1}{4} - \frac{A}{2\alpha} + \frac{B}{2\alpha} \qquad \text{and} \qquad r = \frac{1}{4} - \frac{A}{2\alpha} - \frac{B}{2\alpha}. \tag{51}$$

Substituting the above expressions into equation (11), and using the mapping coordinate transformation (42) and wavefunction transformation (3), we obtain the eigenfunctions of the Scarf I potential,

$$\tilde{\Psi}_n(\theta) = \left(\frac{1 - \sin\alpha\theta}{2}\right)^{\frac{A}{2\alpha} - \frac{B}{2\alpha}} \left(\frac{1 + \sin\alpha\theta}{2}\right)^{\frac{A}{2\alpha} + \frac{B}{2\alpha}} P_n^{-\frac{1}{2} + \frac{A}{\alpha} - \frac{B}{\alpha}, -\frac{1}{2} + \frac{A}{\alpha} + \frac{B}{\alpha}}(\sin\alpha\theta). \tag{52}$$

To the best of our knowledge, this result given in equation (52) has not been reported earlier in the literature.

4.4. *Trigonometric-type generalized Pöschl–Teller potential*

Under the condition $0 \leqslant \alpha\theta \leqslant \pi$, we use the transformation $x \to \theta$ through a mapping function

$$x \equiv f(\theta) = \frac{1}{\alpha}\ln\left(\frac{1 - \cos\alpha\vartheta}{\sin\alpha\theta}\right), \tag{53}$$

and choose $Q_3 = 0, q = 1$. In this case, FPEPM expressed in equation (6) becomes

$$V(x) = -\frac{g}{2}\cos\alpha\theta - \gamma\sin^2\alpha\theta. \tag{54}$$

Substituting equations (53) and (54) into (5), we obtain

$$\tilde{V} - \tilde{E} = -\gamma - \frac{\alpha^2}{4} + \left(-E - \frac{\alpha^2}{4}\right)\mathrm{cosec}^2\alpha\theta + \frac{g}{2}\mathrm{cosec}\,\alpha\theta\cot\alpha\theta. \tag{55}$$

For the trigonometric-type generalized Pöschl–Teller potential, we write $\tilde{V}$ in the form

$$\tilde{V} = -A^2 + (A^2 + B^2 - \alpha A)\,\mathrm{cosec}^2\,\alpha\theta - B(2A - \alpha)\,\mathrm{cosec}\,\alpha\theta\,\cot\alpha\theta, \tag{56}$$

where $A > B$. Comparing equations (55) and (56), we have

$$-\gamma - \frac{\alpha^2}{4} + \tilde{E} = -A^2, \tag{57}$$

$$-E - \frac{\alpha^2}{4} = A^2 + B^2 - \alpha A, \tag{58}$$

$$\frac{g}{2} = -B(2A - \alpha). \tag{59}$$

Using the energy spectrum expression (10), solving equations (58) and (59) yields

$$\gamma = (A + n\alpha)^2 - \frac{\alpha^2}{4}. \tag{60}$$

Substituting (60) into (57), we obtain the energy spectra for the trigonometric-type generalized Pöschl–Teller potential,

$$\tilde{E}_n = -A^2 + (A + n\alpha)^2. \tag{61}$$

Here, we recover the energy spectrum expression presented by De *et al* in table 2 of [4]. From equations (12) and (13), we obtain

$$p = \frac{1}{4} - \frac{A}{2\alpha} + \frac{B}{2\alpha} \qquad \text{and} \qquad r = \frac{1}{4} - \frac{A}{2\alpha} - \frac{B}{2\alpha}. \tag{62}$$

With the help of equations (3), (53) and (61), we obtain the eigenfunctions of the trigonometric-type generalized Pöschl–Teller potential,

$$\tilde{\Psi}_n(\theta) = \left(\frac{1 - \cos\alpha\theta}{2}\right)^{\frac{A-B}{2\alpha}} \left(\frac{1 + \cos\alpha\theta}{2}\right)^{\frac{A+B}{2\alpha}} P_n^{\frac{A}{\alpha} - \frac{B}{\alpha} - \frac{1}{2},\, \frac{A}{\alpha} + \frac{B}{\alpha} - \frac{1}{2}}(\cos\alpha\theta). \tag{63}$$

This result coincides completely with the result given by De *et al* in table 2 of [4].

4.5. Pöschl–Teller I potential

In order to construct the Pöschl–Teller I potential, we choose $Q_3 = 0$, $q = 1$ and introduce a mapping function

$$x \equiv f(\theta) = \frac{1}{\alpha}\ln(\tan\alpha\theta), \tag{64}$$

where $0 \leqslant \alpha\theta \leqslant \frac{\pi}{2}$. Substituting x into (6), FPEPM reads

$$V(x) = \frac{g}{2}(2\sin^2\alpha\theta - 1) - 4\gamma\sin^2\alpha\theta\cos^2\alpha\theta. \tag{65}$$

Substituting equations (64) and (65) into (5), we obtain

$$\tilde{V} - \tilde{E} = -\alpha^2 - 4\gamma + \left(-\frac{g}{2} - E - \frac{\alpha^2}{4}\right)\sec^2\alpha\theta + \left(\frac{g}{2} - E - \frac{\alpha^2}{4}\right)\mathrm{cosec}^2\,\alpha\theta. \tag{66}$$

For the Pöschl–Teller I potential, we write $\tilde{V}$ in the form

$$\tilde{V} = -(A + B)^2 + A(A - \alpha)\sec^2\alpha\theta + B(B - \alpha)\,\mathrm{cosec}^2\,\alpha\theta. \tag{67}$$

Comparing equations (66) and (67), we have

$$-\alpha^2 - 4\gamma + \tilde{E} = -(A+B)^2, \tag{68}$$

$$-\frac{g}{2} - E - \frac{\alpha^2}{4} = A(A-\alpha), \tag{69}$$

$$\frac{g}{2} - E - \frac{\alpha^2}{4} = B(B-\alpha). \tag{70}$$

Substituting expression (10) into equations (69) and (70), we obtain

$$g = B^2 - \alpha B - A^2 + \alpha A, \tag{71}$$

$$\gamma = \frac{1}{4}(A+B+2n\alpha)^2 - \frac{\alpha^2}{4}. \tag{72}$$

Substituting (72) into (68), we obtain the energy spectra for the Pöschl–Teller I potential,

$$\tilde{E}_n = -(A+B)^2 + (A+B+2n\alpha)^2. \tag{73}$$

This result coincides completely with the result given by De *et al* in table 2 of [4]. Substituting equations (71) and (72) into equations (12) and (13), we obtain

$$p = \frac{1}{4} - \frac{B}{2\alpha} \qquad \text{and} \qquad r = \frac{1}{4} - \frac{A}{2\alpha}. \tag{74}$$

Considering the wavefunction transformation (3) and coordinate transformation (64), we obtain the eigenfunctions of the Pöschl–Teller I potential by substituting equation (74) into (11),

$$\tilde{\Psi}_n(\theta) = \left(\frac{1-y}{2}\right)^{\frac{B}{2\alpha}} \left(\frac{1+y}{2}\right)^{\frac{A}{2\alpha}} P_n^{\frac{B}{\alpha}-\frac{1}{2}, \frac{A}{\alpha}-\frac{1}{2}}(y), \tag{75}$$

where $y = 1 - 2\sin^2\alpha\theta$. This result of (75) coincides completely with the result given by De *et al* in table 2 of [4].

5. Conclusions

In the present work, we choose special mapping functions for the FPEPM via PCT, and construct the Rosen–Morse I, trigonometric-type Eckart, Scarf I, trigonometric-type generalized Pöschl–Teller and Pöschl–Teller I potentials. The energy spectra and wavefunctions for the trigonometric-type potentials are obtained in a unified manner by using the expressions for the energy spectra and wavefunctions of the FPEPM. If we choose a proper mapping function for the coordinate transformation, and take FPEPM as input, we can also obtain other exactly solvable potentials via PCT.

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